5. N. T. Bendik, S. K. Smirnov, and E. (Ye.) L. Blinkov, "Calculation of stationary heat transfer zone in conduits under cooldown," Cryogenics, 8, 477-482 (1979).
6. R. B. Scott, Cryogenic Engineering, Van Nostrand, Princeton, New Jersey (1959).
7. R. D. McCarty, "Thermophysical properties of helium-4 from 2 to 1500°K with pressures to 1000 atm," Nat. Bur. Stand. (U.S.), Tech. Note 631 (1972), pp. 78-79.
8. N. T. Bendik and N. E. Komissarzhevskii, "Determination of the temperature of a section of core under nonsteady conditions of operation of a superconducting power cable," Informénergo, B. U. (Vses. Inst. Nauch. Tekh. Inform.), Deposited manuscripts, No. 2, 69 (1979).

CALCULATION OF INITIAL STAGE OF HEATING A PLANAR BODY WITH VARIABLE PROPERTIES

Yu. V. Vidin

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A method is presented for calculation of upper and lower limits of the temperature field of a planar body with temperature-dependent thermophysical properties.

In the initial stage of heating, a planar body can be considered as semiinfinite. The differential transfer equation with consideration of temperature dependence of the thermal conductivity and specific heat can then be written in the form

$$\frac{d}{d\eta}\left[f_1(\Theta)\frac{d\Theta}{d\eta}\right]+2\eta f_2(\Theta)\frac{d\Theta}{d\eta}=0, \tag{1}$$

where $\eta = \sqrt{x^2/a_0\tau}/2$ is the Boltzmann variable, and $f_1(\Theta)$ and $f_2(\Theta)$ are positive functions which do not go to zero over the range of Θ from 0 to 1. We supplement Eq. (1) by the boundary conditions

$$\Theta=0 \quad \text{for} \quad \eta=0, \tag{2}$$

$$\Theta=1 \quad \text{for} \quad \eta\to\infty. \tag{3}$$

In the general case Eq. (1) is nonlinear, so achievement of an analytical solution is difficult.

To study the problem presented by Eqs. (1)-(3), we will use the approach proposed in [1, 2], which considered nonstationary thermal conductivity of bodies with nonlinear boundary conditions. Following [1, 2], we will find upper and lower limits for the unknown temperature field Θ. This method is applicable in engineering practice when the "gap" between the limiting functions is relatively small and the equations involved are relatively simple.

We will now demonstrate the application of this principle to Eqs. (1)-(3).

Introducing the Kirchhoff substitution

$$U=\int\limits_0^\Theta f_1(\Theta)\,d\Theta \Big/ \int\limits_0^1 f_1(\Theta)\,d\Theta, \tag{4}$$

we transform Eqs. (1)-(3) to the form

$$\frac{d^2U}{d\eta^2}+2\eta\frac{f_2(\Theta)}{f_1(\Theta)}\frac{dU}{d\eta}=0, \tag{5}$$

$$U=0 \quad \text{for} \quad \eta=0, \tag{6}$$

$$U=1 \quad \text{for} \quad \eta\to\infty. \tag{7}$$

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Without limiting the generality of the method, we will consider a special case of Eqs. (5)-(7), viz.: $f_1(\Theta) = 1 + \beta\Theta$ and $f_2(\Theta) = 1$, which demonstrates the peculiarities of the solution more clearly.

If the parameter $\beta > 0$, then the lower limit for the temperature field can be established by integration of the linear equation

$$\frac{d^2U_{min}}{d\eta^2} + \frac{2\eta}{1+\beta}\,\frac{dU_{min}}{d\eta} = 0. \tag{8}$$

Solving Eq. (8) with conditions (6), (7), we have [3]

$$U_{min} = \operatorname{erf}\frac{\eta}{\sqrt{1+\beta}}, \tag{9}$$

where $U_{min} = (\Theta_{min} + (\beta/2)\Theta^2_{min})/(1 + \beta/2)$.

The function $\Theta_{min} = \Theta_{min}(\eta)$ found in this manner serves as the basis for calculation of an upper temperature limit. Now we must integrate the equation

$$\frac{d^2U_{max}}{d\eta^2} + \frac{2\eta}{1+\beta\Theta_{min}}\,\frac{dU_{max}}{d\eta} = 0. \tag{10}$$

However, in view of the mathematical complexity of the expression $2\eta/[1 + \beta\Theta_{min}(\eta)]$, it is difficult to obtain a convenient engineering solution directly from Eq. (10). Therefore, in place of Eq. (10) we introduce the expression

$$\frac{d^2U_{max}}{d\eta^2} + \left(a + \frac{2\eta}{1+\beta}\right)\frac{dU_{max}}{d\eta} = 0, \tag{11}$$

where a is a constant subject to determination. We note especially that the condition

$$\left(a + \frac{2\eta}{1+\beta}\right) \geqslant \frac{2\eta}{1+\beta\Theta_{min}(\eta)} \tag{12}$$

must be satisfied over the entire range of η (from 0 to ∞). The coefficient a can be determined most simply by a graphical method, using the construction shown in Fig. 1.

Integrating Eq. (11) together with Eqs. (6), (7), we obtain

$$U_{max} = \frac{\operatorname{erf}\left(\frac{\eta}{\sqrt{1+\beta}} + \frac{a}{2}\sqrt{1+\beta}\right) - \operatorname{erf}\left(\frac{a}{2}\sqrt{1+\beta}\right)}{1 - \operatorname{erf}\left(\frac{a}{2}\sqrt{1+\beta}\right)} \tag{13}$$

and then use the relationship $U_{max} = (\Theta_{max} + (\beta/2)\Theta_{max})/(1+\beta/2)$ to calculate the upper limit $\Theta_{max} = \Theta_{max}(\eta)$. The true function $\Theta = \Theta(\eta)$ is located between Θ_{min} and Θ_{max}, i.e., $\Theta_{min}(\eta) \leqslant \Theta(\eta) \leqslant \Theta_{max}(\eta)$, where the sign = refers to $\eta = 0$ and $\eta \to \infty$. For $\beta = 0$ the upper and lower curves coincide and transform to an exact analytical solution of the problem.

Because the "gap" between Θ_{min} and Θ_{max} for any value of β (from 0 to ∞) proves to be narrow, subsequent refinement of Θ_{min} and Θ_{max} is unnecessary.

Figure 2 shows a calculation of the lower and upper limits using Eqs. (9) and (13) for the case $\beta = 3$. Also shown is the actual temperature [4]. It is evident from the figure that the difference between Θ_{min} and Θ_{max} is small, and that the two functions are located almost symmetrically with respect to $\Theta = \Theta(\eta)$.

If $\beta < 0$, then Eq. (9) gives the maximum value U_{max}, i.e., then

$$U_{max} = \operatorname{erf}\frac{\eta}{\sqrt{1+\beta}}. \tag{14}$$

To find the lower limit we must integrate the equation

$$\frac{d^2U_{min}}{d\eta^2} + \frac{2\eta}{1+\beta\Theta'_{min}}\,\frac{dU_{min}}{d\eta} = 0, \tag{15}$$

where by Θ'_{min} we understand the solution of the equation

$$U'_{min} = \left[\Theta'_{min} + \frac{\beta}{2}(\Theta'_{min})^2\right]\Big/\left(1 + \frac{\beta}{2}\right) = \operatorname{erf}\eta. \tag{16}$$

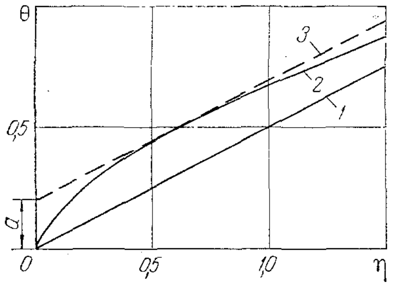

Fig. 1

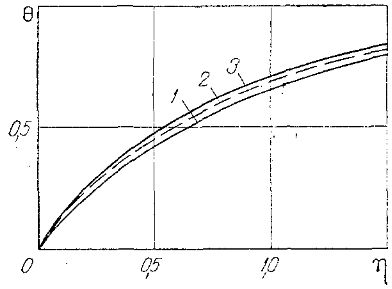

Fig. 2

Fig. 1. Graphical method of determining coefficient a ($\beta = 3$): 1) $2\eta/(1+\beta)$; 2) function $2\eta/[1+\beta\Theta_{min}(\eta)]$; 3) tangent to curve 2, parallel to curve 1.

Fig. 2. Lower (1) and upper (3) limits for temperature field Θ (2) for $\beta = 3$.

In analogy with Eq. (10), instead of Eq. (15) we have

$$\frac{d^2U_{min}}{d\eta^2} + \left(a + \frac{2\eta}{1+\beta}\right)\frac{dU_{min}}{d\eta} = 0. \tag{17}$$

The inequality sign of Eq. (12) then changes its direction:

$$\left(a + \frac{2\eta}{1+\beta}\right) \leqslant \frac{2\eta}{1+\beta\Theta'_{min}(\eta)}. \tag{18}$$

In conclusion, we note that with no changes in principle this approach can be applied to the more general case described by Eq. (5).

LITERATURE CITED

1. Yu. V. Vidin, "Nonstationary temperature field in a solid with nonlinear boundary conditions," Inzh.-Fiz. Zh., 19, No. 2, 355-356 (1970).
2. Yu. V. Vidin, "Temperature regime of a massive multilayer lamina, simultaneously heated by convection and radiation," Izv. Akad. Nauk SSSR, Energ. Transport, No. 1, 181-184 (1970).
3. A. V. Lykov, Theory of Thermal Conductivity [in Russian], Vysshaya Shkola, Moscow (1967).
4. S. Sh. Tsoi and Ya. E. Sanderlend, "Thermal conductivity problems with phase transitions and temperature dependence of the thermal conductivity coefficient," Teploperedacha, No. 2, 110-114 (1974).